EC202C1 – Intermediate Macroeconomic Analysis
Spring 2012, Boston University

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Second Mid-term Exam (Practice #2)

Friday, April 6, 2012

This is a 50-minute exam. There is a total of 50 points allocated across two questions. Use the number of points allocated to each part as a suggestion for how long to spend on that part. I recommend that you attempt all parts before using more time than is suggested for any one part. If you complete some parts in less than the suggested time, use your extra time to revisit parts you may have had trouble with the first time through and to check your work.

Please read the questions carefully and write your answers in the space provided. You can use the backs of the sheets for scrap paper, but to get full credit you must show all relevant work in the space provided.

Please follow my instructions at all times.

Concentrate and think carefully, but try to relax too!

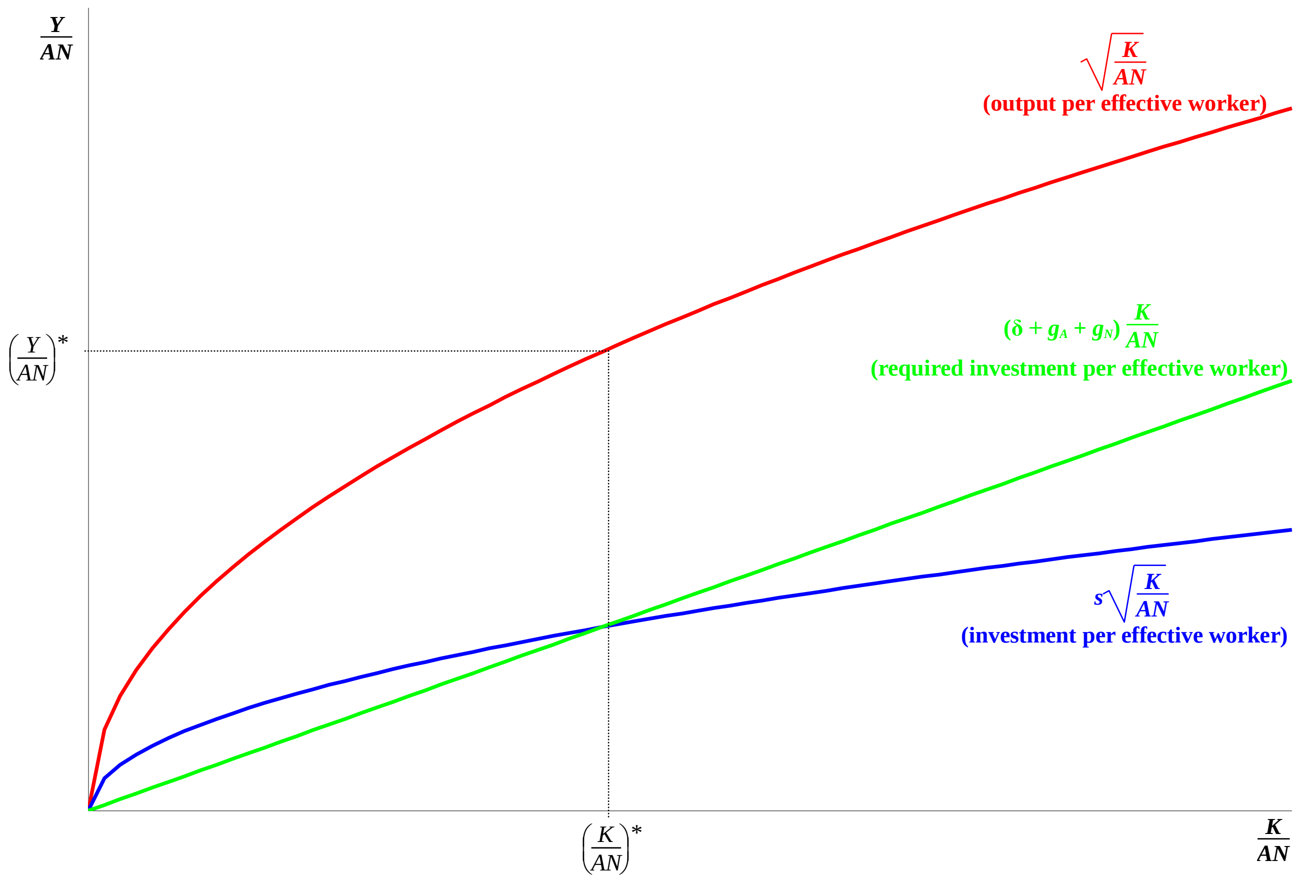
Student Number: Solutions

(Please do not include your name.)

1. [18 points total, 2 parts] Consider the full Solow growth model with exogenous technological progress and employment growth. The economy is described by the production function $Y = \sqrt{K}\sqrt{AN}$. Technology, A, grows at the fixed percentage rate g_A per period, and employment, N, grows at the fixed percentage rate g_N per period. The capital stock, K, depreciates at the fixed rate δ per period, and the economy saves a fixed proportion, s, of output, Y, per period. Assume throughout that taxes and government expenditure are zero. You can assume that the symbolic representation of the necessary investment function is given by $(\delta + g_A + g_N)K$ and that the symbolic representation of the intensive form of the production function is $\sqrt{\frac{K}{AN}}$ without any derivation.

a) [10 points] Draw a graph showing the output per effective worker, investment per effective worker, and necessary investment per effective worker functions. Label each of these three curves with their corresponding symbolic representations. Identify the steady state levels of capital per effective worker and output per effective worker (but you do not need to derive their expressions).

answer:



(It's fine to use "necessary investment" and "required investment" synonymously. The question asks for the necessary investment per effective worker function, but the graph calls it the required investment per effective worker function. It's the same thing.)

b) [8 points] State the rate at which each of the following variables grows in the steady state.

Consumption	______ $g_A + g_N$ _____
Consumption per worker	________ g_A _______
Consumption per effective worker	_________0________
Output per worker	________ g_A _______
Required investment	______ $g_A + g_N$ _____
The depreciation rate	_________0________
Saving per effective worker	_________0________
Output per unit of capital	_________0________

As was also suggested in the first practice exam, you should confirm these answers by reasoning your way to them using the tools that you've learned. Don't just try to identify patterns to memorize.

As an example, take the last one, which is unfamiliar but not necessarily difficult. Output per unit of capital, or Y/K, will grow at the growth rate of output minus the growth rate of the capital stock, by one of the basic growth rate rules we know. In the steady state, output will grow at the growth rate of technology plus the growth rate of employment (from noting that output per effective worker is constant in the steady state and applying the growth rate rules as we've done before, which you should also take this opportunity to refresh yourself on). In the steady state, the capital stock will also grow at the rate of growth of technology plus the rate of growth of employment (by the same growth rate rules and chain of reasoning). Therefore, the numerator of Y/K will be growing at the same rate as the denominator is growing in the steady state, so that the ratio itself will be constant in the steady state (or, in other words, the growth rate of output minus the growth rate of the capital stock – which we said above gives the growth rate of the ratio in question – is zero in the steady state).

2. [32 points total, 4 parts] Consider the simple Solow model with no technological progress and no employment growth. The economy is described by the production function $Y = \sqrt{K}\sqrt{N}$. Technology is fixed at a level of one, and the number of workers, N, is constant. The capital stock, K, depreciates at the fixed rate of 10% per period, and the economy saves a fixed proportion, s, of output, Y, per period. Assume throughout that taxes and government expenditure are zero.

a) [10 points] Derive the expression for the steady state capital-labor ratio. Show all of your work.

answer:

<u>production function</u> (1)

$$Y = \sqrt{K}\sqrt{N}$$

$$\frac{Y}{N} = \frac{\sqrt{K}\sqrt{N}}{N} = \sqrt{\frac{K}{N}}.$$

<u>investment</u> (2)

saving = sY

and, by equilibrium in the goods market,

investment = saving $\rightarrow I_t = sY_t$.

<u>capital accumulation</u>

$$K_{t+1} = K_t - 0.1K_t + I_t$$

$$\frac{K_{t+1}}{N} = \frac{K_t}{N} - 0.1\frac{K_t}{N} + s\,\frac{Y_t}{N}$$ [substituted (2) and divided through by N]

$$\frac{K_{t+1}}{N} - \frac{K_t}{N} = s\sqrt{\frac{K_t}{N}} - 0.1\frac{K_t}{N}.$$ [substituted (1) and rearranged]

<u>steady state</u>

In the steady state, the capital stock will be constant at some level K^*:

$$\frac{K^*}{N} - \frac{K^*}{N} = s\sqrt{\frac{K^*}{N}} - 0.1\frac{K^*}{N}$$

$$\frac{s}{0.1} = \sqrt{\frac{K^*}{N}} \quad \rightarrow \quad \frac{K^*}{N} = 100s^2.$$

(If numbers are given for some of the parameters, I expect you to use them in the derivation, not to just derive things symbolically by rote. And the words "Derive" and "Show all of your work" in the question clearly indicate that just writing down and/or plugging values into memorized expressions is not satisfactory.)

b) [8 points]

i. Find an expression for the steady state level of consumption per worker. Show your work.

ii. Show that a saving rate of 35% is *not* the "golden rule" saving rate.

answer:

i.

$Y = C + I + G$ [equilibrium condition]

$Y = C + I$ [government expenditure assumed zero]

$C = Y - I$

$C = Y - sY$ [saving = investment in equilibrium, and saving = sY as usual]

$\frac{C^*}{N} = (1 - s)\frac{Y^*}{N}$ [divide through by N and evaluate all variables at their steady state levels]

$$\frac{C^*}{N} = (1 - s)\sqrt{\frac{K^*}{N}} = (1 - s)\sqrt{100s^2} = 10(s - s^2).$$

[substitute in the intensive form of the production function for output per worker, and then the expression for the steady state capital-labor ratio found in the previous part, and simplify]

ii.

The golden rule saving rate is the saving rate that maximizes steady state consumption per worker. So if 35% were the golden rule saving rate, it wouldn't be possible to find another saving rate associated with a higher level of steady state consumption per worker.

$$\frac{C^*}{N}(0.35) = 10(0.35 - 0.35^2) = 2.275.$$

$$\frac{C^*}{N}(0.45) = 10(0.45 - 0.45^2) = 2.475.$$

These two calculations prove that 35% cannot be the golden rule saving rate. The arbitrarily chosen 45% saving rate is associated with higher steady state consumption per worker than the candidate 35% saving rate is. Therefore, the candidate 35% saving rate violates the definition of the golden rule saving rate. (Note that this part does not require you to find the actual golden rule saving rate. That would require calculus. It was fine if you took the first order condition and solved it – and I won't even take points off for forgetting to check the second-order condition – as long as you completed the argument by saying that your answer is not 35%, so 35% cannot maximize consumption per worker and therefore cannot be the golden rule saving rate. But this is overkill, and a much more mechanical than intuitive way of thinking about the situation. All that was required was to show a counter-example, as was done above.)

c) [8 points] Consider five levels of the capital-labor ratio, ordered from smallest to largest: $\left(\frac{K}{N}\right)_h < \left(\frac{K}{N}\right)_i < \frac{K^*}{N} < \left(\frac{K}{N}\right)_j < \left(\frac{K}{N}\right)_k$. As usual, $\frac{K^*}{N}$ refers to the steady state level of the capital-labor ratio. The saving rate is fixed at a single value, s, throughout. Answer the following two multiple-choice questions by clearly indicating, for each, the response that is most accurate.

I. If, in some period t, the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_h$, the change in the capital-labor ratio between periods t and t+1 would be

[i.] positive and larger in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_i$.

ii. positive and smaller in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_i$.

iii. negative and larger in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_i$.

iv. negative and smaller in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_i$.

v. zero.

II. If, in some period t, the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_j$, the change in the capital-labor ratio between periods t and t+1 would be

i. positive and larger in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_k$.

ii. positive and smaller in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_k$.

iii. negative and larger in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_k$.

[iv.] negative and smaller in magnitude than if the capital-labor ratio were at the level $\left(\frac{K}{N}\right)_k$.

v. zero.

See the next part for an extended explanation and accompanying graph.

d) [6 points] Find a general expression for the percentage growth rate of the capital-labor ratio between any two periods t and t+1 from any arbitrary starting point. (This expression should depend on the saving rate and on the level of the capital-labor ratio at time t.) Then note that the capital-labor ratio at time t can be expressed as some proportion v of the steady state level of the capital-labor ratio. Use this information to simplify the expression for the percentage growth rate of the capital-labor ratio. (The expression should now depend on v only.) Does this expression correspond with your answers in the previous part? Explain.

answer:

[Note: I did not make this part worth many points because I was expecting it to be difficult and did not want there to be an overly harsh penalty for not completing it perfectly. I gave this on a mid-term in a previous semester, and the class found it even more difficult than I was expecting. I will not purposely give something so difficult on any exams this semester. But you should nonetheless be able to conceptually understand and digest the following discussion.]

Start with the equation for the difference in the capital-labor ratio across periods that we derived from the capital accumulation equation in part a). Then, divide through by the capital-labor ratio in time t to calculate the percentage change.

$$\frac{K_{t+1}}{N} - \frac{K_t}{N} = s\sqrt{\frac{K_t}{N}} - 0.1\frac{K_t}{N}$$

$$g_{K/N} \equiv \frac{\frac{K_{t+1}}{N} - \frac{K_t}{N}}{\frac{K_t}{N}} = \frac{s\sqrt{\frac{K_t}{N}} - 0.1\frac{K_t}{N}}{\frac{K_t}{N}} = s / \sqrt{\frac{K_t}{N}} - 0.1.$$

Now set $\frac{K_t}{N} = v\,\frac{K^*}{N} = z100s^2$:

$$g_{K/N} = s / \sqrt{v100s^2} - 0.1 = s/10s\sqrt{v} - 0.1 = 0.1(1/\sqrt{v} - 1).$$

You can think of this equation as representing the speed of convergence to the steady state as a function of the distance of the economy's starting point from the steady state. It encapsulates a number of nice pieces of intuition concerning the overall model.

First, note that, when v equals 1, $g_{K/N}$ collapses to zero. That is, when the economy starts at a capital-labor ratio that is 100% of the steady state capital-labor ratio, the growth rate of the capital-labor ratio is 0%. Or in other words, when the economy is in its steady state, the capital-labor ratio will not be growing – which is the definition of the steady state and a fundamental observation concerning the model.

Second, when v is greater than 1 (and hence the capital-labor ratio is more than 100% of the steady state level), the first term in parentheses is less than 1, so $g_{K/N}$ is negative. That is, when the capital-labor ratio is above its steady state level, the capital-labor ratio will shrink over time.

When *v* is less than 1 (and hence the capital-labor ratio is less than 100% of the steady state level), the first term in parentheses is greater than 1, so $g_{K/N}$ is positive. That is, when the capital-labor ratio is below its steady state level, the capital-labor ratio will grow over time. In other words, when the economy is away from its steady state in either direction, the growth dynamics inherent in the model are such that the economy will be moving back towards (i.e. converging to) its steady state.

Third, the equation for $g_{K/N}$ is uniformly decreasing in *v*. When the capital-labor ratio is very far below its steady state level (i.e. *v* is very small), it will be growing very rapidly. When the capital-labor ratio is only slightly below its steady state level (i.e. *v* is less than but close to 1), it will be growing slowly. When the capital-labor ratio is slightly above its steady state level (i.e. *v* is greater than but close to 1), it will be decreasing slowly. When the capital-labor ratio is very far above its steady state level (i.e. *v* is very large), it will be decreasing rapidly. In other words, the further the economy is from its steady state, the more rapidly will it be converging towards its steady state, and as it gets nearer to its steady state, the rate of convergence will diminish. This is due to the combination of diminishing marginal returns to capital (which are responsible for the curvature of the production function) and the linear increase in depreciation per worker with increases in the capital-labor ratio.

These properties of the growth rate equation are precisely mirrored in the multiple-choice questions of the previous part. Point *h* in that set-up was very far below the steady state, and hence, according to the growth rate equation, should be associated with positive and rapid growth. Looking at it on a graph, we can see that investment per worker is far exceeding depreciation per worker in this range, so that there will indeed be a net increase in the capital-labor ratio, and it will indeed be a relatively large net increase. Point *j* in that set-up was slightly above the steady state, and hence, according to the growth rate equation, should be associated with negative and slow growth. Looking at it on a graph, we can see that depreciation per worker is slightly above investment per worker in this range, so that there will indeed be a net decrease in the capital-labor ratio, and it will indeed be a relatively small net decrease.