

Practice Problem Set #5 – Solutions

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1. Consider a simple version of the Solow growth model, with no technology growth and no growth in employment. Suppose that the economy is described by the production function $Y = A\sqrt{K}\sqrt{N}$, where Y is output, K is the level of the capital stock, N is the (constant) number of workers, and A is the (constant) level of technology. Assume throughout that taxes and government expenditure are zero.

a) Convert the production function to intensive form (i.e. find an expression for output per worker).

answer:

See the margin notes on page 214 of the custom-printed version of the textbook and the main discussion on page 226. This is essentially the same specification of the production function, except that here we've explicitly included technology with the constant multiplicative factor A , which doesn't change the mechanics.

$$Y = A\sqrt{K}\sqrt{N}$$

$$\frac{Y}{N} = \frac{A\sqrt{K}\sqrt{N}}{N} \quad \text{[divided both sides by } N\text{]}$$

$$\frac{Y}{N} = \frac{A\sqrt{K}}{\sqrt{N}} \quad \text{[because } \sqrt{N}/N = 1/\sqrt{N}\text{]}$$

$$\frac{Y}{N} = A\sqrt{\frac{K}{N}}. \quad \text{[This is what we refer to generally as } f\left(\frac{K}{N}\right)\text{.]}$$

b) Write down the standard equation describing how the capital stock evolves over time, using δ for the depreciation rate. From this, derive an expression for the change in the capital-labor ratio between years t and $t+1$ that depends only on the level of the capital-labor ratio in year t and parameters, using s for the saving rate. Explain your steps.

answer:

See page 216 in the text. Generally, the capital stock next period will be equal to the capital stock this period, minus that portion of it that has depreciated away, plus the new capital that has been purchased in the form of investment. Putting this into a symbolic mathematical expression,

$$K_{t+1} = K_t - \delta K_t + I_t$$

$$K_{t+1} = K_t - \delta K_t + sY_t \quad \text{[because investment is equal to saving for the goods market to be in equilibrium, and we assume that saving is a fixed proportion of output in each period, as usual]}$$

$$\frac{K_{t+1}}{N} = \frac{K_t}{N} - \delta \frac{K_t}{N} + s \frac{Y_t}{N} \quad \text{[divided each side by } N \text{]}$$

$$\frac{K_{t+1}}{N} - \frac{K_t}{N} = sA\sqrt{\frac{K_t}{N}} - \delta \frac{K_t}{N}. \quad \text{[inserted the intensive form of the production function for output per worker, and rearranged terms]}$$

c) Find an expression for the steady state capital-labor ratio and the steady state level of output per worker. Show your work.

answer:

In the steady state, the capital stock will be constant at some level K^* . Insert this into the rearranged capital accumulation equation just derived:

$$\frac{K^*}{N} - \frac{K^*}{N} = sA\sqrt{\frac{K^*}{N}} - \delta \frac{K^*}{N}$$

$$sA\sqrt{\frac{K^*}{N}} = \delta \frac{K^*}{N} \quad \text{[rearranging, noting that the left-hand side above is zero]}$$

$$\frac{sA}{\delta} = \frac{K^*}{N} / \sqrt{\frac{K^*}{N}} \quad \text{[dividing both sides by } \delta\sqrt{\frac{K^*}{N}} \text{]}$$

$$\sqrt{\frac{K^*}{N}} = \frac{sA}{\delta} \quad \text{[rearranging, and because } a / \sqrt{a} = \sqrt{a} \text{ for any arbitrary variable } a \text{]}$$

$$\frac{K^*}{N} = \left(\frac{sA}{\delta}\right)^2. \quad \text{[squaring both sides]}$$

Plug this into the intensive form of the production function to find output per worker in the steady state:

$$\frac{Y^*}{N} = A\sqrt{\frac{K^*}{N}}$$

$$\frac{Y^*}{N} = A\sqrt{\left(\frac{sA}{\delta}\right)^2} \quad \text{[inserting the expression for } \frac{K^*}{N} \text{ just found]}$$

$$\frac{Y^*}{N} = A \left(\frac{sA}{\delta} \right) \quad \text{[evaluating the square root of the square]}$$

$$\frac{Y^*}{N} = A^2 \left(\frac{s}{\delta} \right). \quad \text{[simplifying]}$$

d) Find an expression for the steady state level of investment per worker and the steady state level of depreciation per worker. Do your answers make sense in relation to each other? Explain.

answer:

Investment per worker:

$$\frac{I^*}{N} = s \frac{Y^*}{N} \quad \text{[start with the investment equals saving condition, divide both sides by } N, \text{ insert the usual saving assumption, and evaluate all variables at their steady state values]}$$

$$\frac{I^*}{N} = sA^2 \left(\frac{s}{\delta} \right) \quad \text{[inserting the expression for } \frac{Y^*}{N} \text{ found above]}$$

$$\frac{I^*}{N} = \frac{(sA)^2}{\delta}. \quad \text{[simplifying]}$$

Depreciation per worker:

$$\begin{aligned} \text{steady state depreciation per worker} &= \delta \frac{K^*}{N} \\ &= \delta \left(\frac{sA}{\delta} \right)^2 \quad \text{[inserting the expression for } \frac{K^*}{N} \text{ found above]} \end{aligned}$$

$$= \frac{(sA)^2}{\delta}. \quad \text{[simplifying]}$$

So depreciation per worker and investment per worker are equal to the same value in the steady state. This is, in fact, the very definition of the steady state: the capital-labor ratio will be constant across periods precisely when investment per worker is just enough to exactly offset depreciation per worker. This is therefore what we would expect to find.

e) Find an expression for the steady state level of consumption per worker.

answer:

There are a number of different ways to find this using the results that have already been derived. For each, we need to start with the condition for goods market equilibrium:

$$Y = C + I + G$$

$$Y = C + I \quad [\text{because government expenditure is zero by assumption}]$$

$$C = Y - I$$

The simplest way at this point is to substitute in the investment function under the usual assumptions, divide by the number of workers, and evaluate everything at steady state values:

$$C = Y - sY \quad [\text{by our usual assumptions concerning investment}]$$

$$\frac{C^*}{N} = (1 - s) \frac{Y^*}{N} \quad [\text{simplifying, dividing both sides by } N, \text{ and evaluating variables at their steady state values}]$$

$$\frac{C^*}{N} = (1 - s) A^2 \left(\frac{s}{\delta} \right) \quad [\text{inserting the expression for } \frac{Y^*}{N} \text{ found above}]$$

$$\frac{C^*}{N} = \left(\frac{A^2}{\delta} \right) (s - s^2). \quad [\text{simplifying}]$$

This negative quadratic form describes the hump-shaped relationship between the saving rate and steady state consumption per worker discussed in class and on pages 222-4 of the book.

2. Consider a simple version of the Solow growth model, with no technology growth and no growth in employment. Suppose that the economy is described by the production function $Y = A\sqrt{K}\sqrt{N}$, where Y is output, K is the level of the capital stock, N is the (constant) number of workers, and A is the level of technology (which is constant from period to period but can experience one-time changes exogenously). Assume throughout that taxes and government expenditure are zero.

a) Draw a graph showing the intensive form of the production function for some technology level A_1 , the depreciation-per-worker function for some depreciation rate δ_1 , and the investment-per-worker function for some saving rate s_1 . Label the associated steady state capital-labor ratio and steady state level of output per worker.

answer:

See Figure 1.

b) Suppose that the saving rate increases to $s_2 > s_1$ but that the technology level and depreciation rate stay constant at A_1 and δ_1 respectively. Illustrate the initial steady state and the new steady state in a graph without the production function. Describe briefly how the economy will converge to the new steady state.

answer:

See Figure 2.

The new steady state will entail a higher capital-labor ratio compared with the original steady state. If the economy starts in the original steady state, investment per worker will, by definition, be just enough to offset depreciation per worker and hence keep the capital-labor ratio constant. In the period in which the saving rate increases, investment per worker will immediately increase from its steady-state level. (The capital-labor ratio and hence output per worker will stay constant in the instant that the saving rate increases. But investment per worker, which is given by the saving rate times output per worker, will be higher due to the higher saving rate.) The higher investment per worker but unchanged depreciation per worker (because, again, the capital stock does not change immediately) will lead to a higher capital-labor ratio in the next period, and hence a higher level of output per worker. (See the capital accumulation equation and the intensive form of the production function.) The higher level of output per worker will lead to yet higher investment per worker, which in turn will lead to yet a higher capital-labor ratio and higher output per worker, again leading to an increase in investment per worker. At the same time, depreciation per worker is increasing as the capital-labor ratio increases. Indeed, the increase in investment per worker across successive periods will be slightly smaller than the increase in depreciation per worker, due to the curvature of the production and hence investment functions (reflecting diminishing marginal returns to capital) versus the linearity of the depreciation per worker function. This cycle – from investment to capital to output and back to investment – will hence proceed in smaller and smaller steps over time, and will eventually conclude when the size of the step reaches zero, at which point the economy will have converged to its new steady state. In this new long run equilibrium, the higher steady state level of investment per worker will again be exactly enough to offset the higher steady state level of depreciation per worker and hence keep the capital-labor ratio constant at its new, higher steady state level from period to period.

Note that this model is not capable of accounting for the short-run recessionary effect of increasing saving (or, in other words, decreasing autonomous consumption) that we studied at the beginning of the semester. Essentially, we have to imagine that each “period” in this model encompasses medium-run adjustments following such short-run movements. This model can only describe what happens across these periods, and is not fine-tuned enough to describe what happens within one of these periods.

c) Forget about the change in the saving rate. Suppose that the depreciation rate rises to $\delta_2 > \delta_1$ but that the technology level and saving rate stay constant at A_1 and s_1 respectively. Illustrate the initial steady state and the new steady state in a graph without the production function. Describe briefly how the economy will converge to the new steady state.

answer:

See Figure 3.

The new steady state will entail a lower capital-labor ratio compared with the original steady state. If the economy starts in the original steady state, investment per worker will, by definition, be just enough to offset depreciation per worker and hence keep the capital-labor ratio constant. In the period in which the depreciation rate increases, depreciation per worker will immediately increase from its steady-state level. (The capital-labor ratio and hence output per worker will stay constant in the instant that the depreciation rate increases. But depreciation per worker, which is given by the depreciation rate times the capital-labor ratio, will be higher due to the higher depreciation rate.) The higher depreciation per worker but unchanged investment per worker (because, again, the capital-labor ratio and hence output per worker do not change immediately, and investment per worker is equal to the saving rate times output per worker) will lead to a lower capital-labor ratio in the next period, and hence a lower level of output per worker. (See the capital accumulation equation and the intensive form of the production function.) The lower level of output per worker will lead to lower investment per worker, which in turn will lead to yet a lower capital-labor ratio and lower output per worker, again leading to a decrease in investment per worker. At the same time, depreciation per worker is decreasing as the capital-labor ratio decreases. Indeed, the decrease in investment per worker across successive periods will be slightly smaller than the decrease in depreciation per worker, due to the curvature of the production and hence investment functions (reflecting diminishing marginal returns to capital) versus the linearity of the depreciation per worker function. This cycle – from capital to output to investment and back to capital – will hence proceed in smaller and smaller steps over time, and will eventually conclude when the size of the step reaches zero, at which point the economy will have converged to its new steady state. In this new long run equilibrium, the lower steady state level of investment per worker will again be exactly enough to offset the lower steady state level of depreciation per worker and hence keep the capital-labor ratio constant at its new, lower steady state level from period to period.

d) Forget about the change in the depreciation rate. Suppose that the technology level rises to $A_2 > A_1$ but that the depreciation rate and saving rate stay constant at δ_1 and s_1 respectively. Illustrate the initial steady state and the new steady state in a graph without the production function. Describe briefly how the economy will converge to the new steady state. (Though similar on the surface to part b), be careful to understand the differences. You might want to draw the graph again including the production function.)

answer:

See Figure 4.

The new steady state will entail a higher capital-labor ratio compared with the original steady state. If the economy starts in the original steady state, investment per worker will, by definition, be just enough to offset depreciation per worker and hence keep the capital-labor ratio constant. In the period in which technology increases, investment per worker will immediately increase from its steady-state level. (The capital-labor ratio will stay constant in the instant that technology increases. But output per worker and hence investment per worker will be higher because output per worker is equal to the higher technology level times the square root of the capital-labor ratio and investment per worker is equal to this times the saving rate.) The higher investment per worker but unchanged depreciation per worker (because, again, the capital stock

does not change immediately) will lead to a higher capital-labor ratio in the next period, and hence a higher level of output per worker. (See the capital accumulation equation and the intensive form of the production function.) The higher level of output per worker will lead to yet higher investment per worker, which in turn will lead to yet a higher capital-labor ratio and higher output per worker, again leading to an increase in investment per worker. At the same time, depreciation per worker is increasing as the capital-labor ratio increases. Indeed, the increase in investment per worker across successive periods will be slightly smaller than the increase in depreciation per worker, due to the curvature of the production and hence investment functions (reflecting diminishing marginal returns to capital) versus the linearity of the depreciation per worker function. This cycle – from investment to capital to output and back to investment – will hence proceed in smaller and smaller steps over time, and will eventually conclude when the size of the step reaches zero, at which point the economy will have converged to its new steady state. In this new long run equilibrium, the higher steady state level of investment per worker will again be exactly enough to offset the higher steady state level of depreciation per worker and hence keep the capital-labor ratio constant at its new, higher steady state level from period to period.

In this scenario, the production function shifts. This is not true in the first two scenarios. Figure 5 shows the impact of the technology shift in more detail: the production function shifts up, which pulls the investment-per-worker function up too, even though the saving rate stays constant. Make sure you understand that the production function does not change throughout parts b) and c). In the case of part d), the initial jump in investment per worker is caused by the constant saving rate multiplying the instantaneously higher level of output per worker following the technology shift. In part b), the initial jump in investment per worker resulted from the higher saving rate multiplying the constant level of output per worker. However, the mechanics of convergence are the same in both parts once the initial jump in investment per worker, whichever the source, has set off the cycle. The graphs are drawn such that the increase in the steady state capital-labor ratio is about the same in parts b) and d). But note that, while output per worker is only a bit higher in the new steady state compared to the old in part b), it is much higher in the new steady state compared to the old in part d), due to the shift in the production function. You might want to combine Figure 1 with Figure 2 in a new graph, then combine Figure 4 with Figure 5 in a separate new graph, and explore the differences visually.

e) State how the level of output per worker in the new steady state compares with that in the initial steady state for each of these three scenarios. Make sure this accords both with your intuition and with the algebraic expression you found for steady state output per worker in the first problem.

answer:

In part b), it's a bit higher. In part c), it's a bit lower. In part d), it's a lot higher. Revisit the discussion above concerning how part b) differs from part d). All of this accords with the expression found for the steady state level of output per worker near the top of page 3 above: higher s or higher A lead to higher steady state output per worker, reflected by each being in the numerator, while A being squared indicates its stronger effect; and higher δ leads to lower steady state output per worker, reflected by its presence in the denominator.

I had wanted to include some kind of an analogy to help with intuition, but there aren't any entirely satisfactory ones. I'll see what I can do with an old treadmill analogy; it's good for thinking about different steady states in relation to each other, but not really for thinking about convergence between them. Think of a long moving sidewalk, and picture yourself in the middle of it, walking in the *opposite* direction as the sidewalk is moving. If you walk at precisely the same speed as the sidewalk is moving, you'll remain in the same place. This is the steady state. The speed of the sidewalk represents depreciation per worker, and the speed at which you're walking represents investment per worker. Your horizontal position represents the level of output per worker.

Start in the middle of the sidewalk in a steady state, i.e. walking at exactly the same speed as the sidewalk is moving and hence staying in the same horizontal position. Then start walking faster (i.e. increase saving). You will start to make forward progress because you will have begun to walk faster than the sidewalk is moving in the opposing direction. You will continue to make forward progress until someone comes along and increases the speed at which the sidewalk is moving (i.e. increases depreciation). When walking speed is again aligned with sidewalk speed, you'll again be holding a fixed horizontal position, but it will be forward from your previous fixed position. (You can see how the convergence process isn't reflected very well in this analogy. In the model, depreciation per worker increases automatically following an increase in investment per worker due initially to an increase in the saving rate; but in the analogy, we need to imagine some external operator. Also, in the model, both investment and depreciation per worker continue to adjust upward in successive periods as convergence proceeds; trying to reflect this pushes the analogy too far. Finally, remember that the higher steady state level of investment per worker in the scenario of part b) is due to the permanently higher saving rate and the higher steady state level of the capital-labor ratio; while the higher steady state level of depreciation per worker is due only to the higher steady state level of the capital-labor ratio, since the depreciation rate has remained constant by construction.)

Start once again back in the middle of the sidewalk in a steady state, i.e. walking at exactly the same speed as the sidewalk is moving and hence staying in the same horizontal position. Then imagine that the sidewalk controller falls asleep on the controls and accidentally makes the sidewalk start to move faster (i.e. depreciation increases). You will start to make backward progress because the sidewalk will have begun to move more rapidly than you are walking in the opposing direction. Then imagine that you start to walk more slowly (i.e. decrease saving) while at the same time, the operator wakes up and reduces the speed of the sidewalk to a pace even lower than its initial pace. When sidewalk speed is again aligned with walking speed, you'll again be holding a fixed horizontal position, but it will be behind your previous fixed position. (Again, this doesn't capture the convergence process very well at all. Note especially that the lower steady state level of depreciation per worker in the scenario of part c) is due to the lower steady state level of the capital-labor ratio, which is actually enough to offset the permanently *higher* depreciation rate; while the lower steady state level of investment per worker is due only to the lower steady state level of the capital-labor ratio, since the saving rate has remained constant by construction. In any case, these two silly stories are just meant to help you visualize how the new steady state will relate to the old steady state after a change in the saving rate or depreciation rate respectively and after the convergence process has completed.)

Figure 1

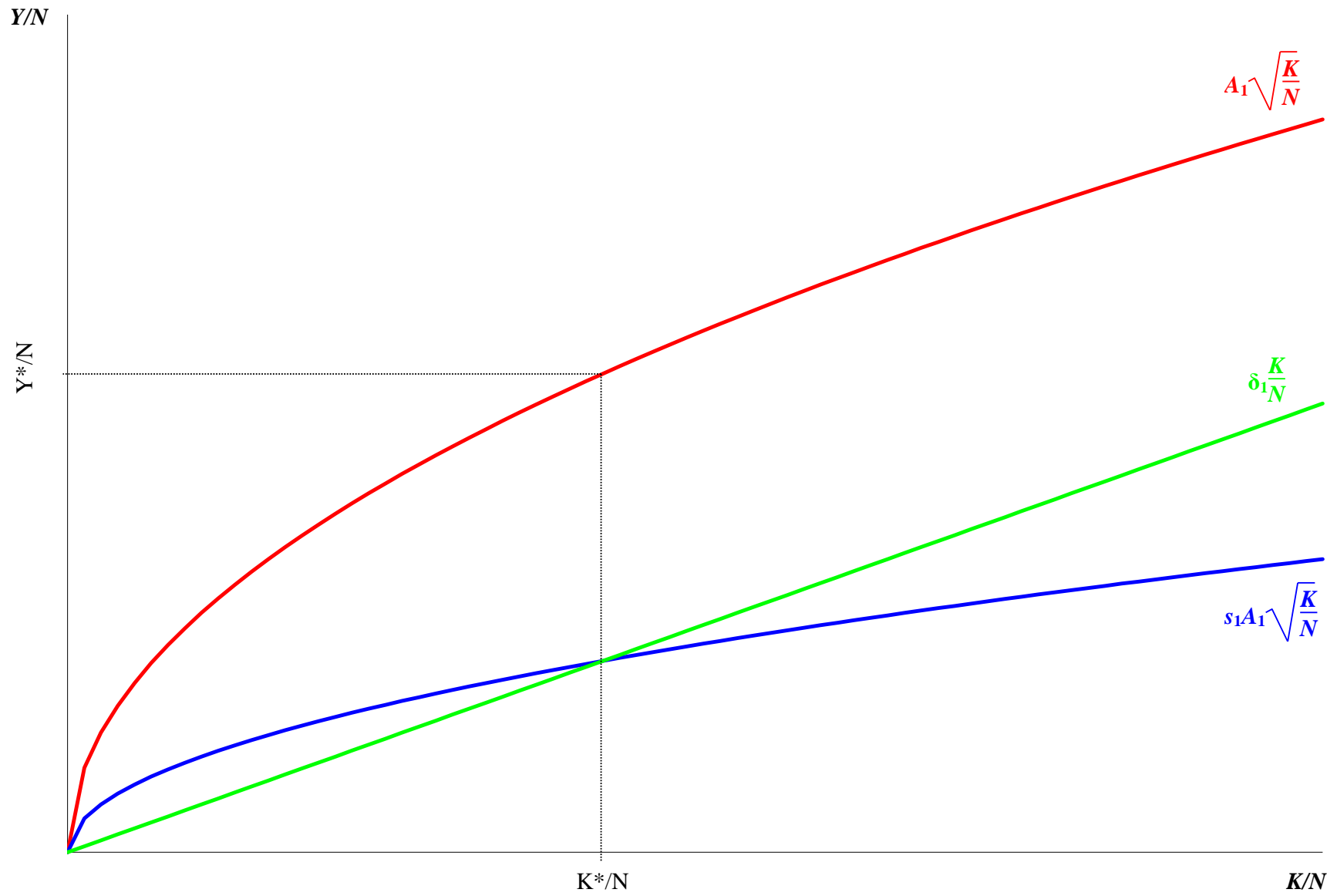


Figure 2

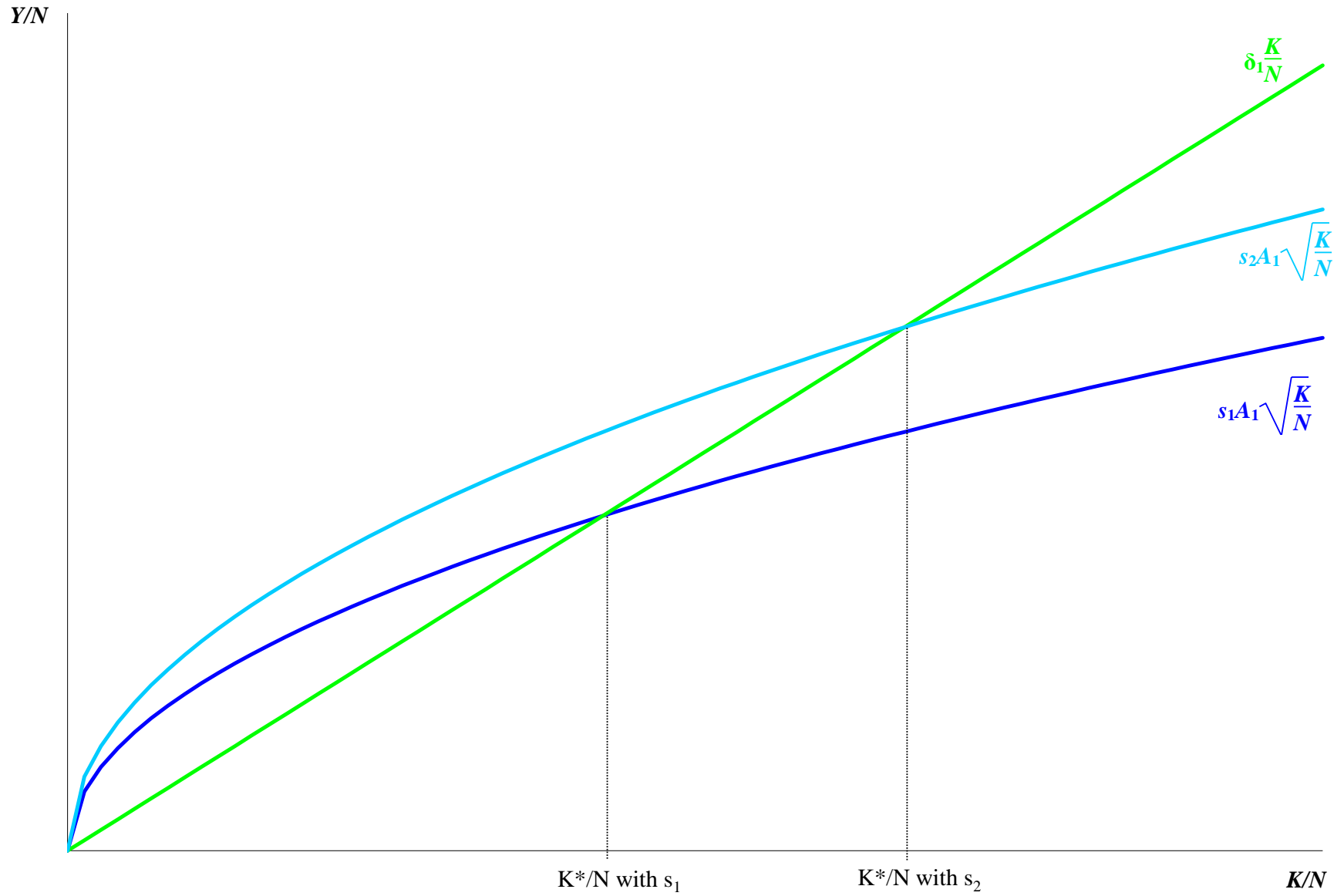


Figure 3

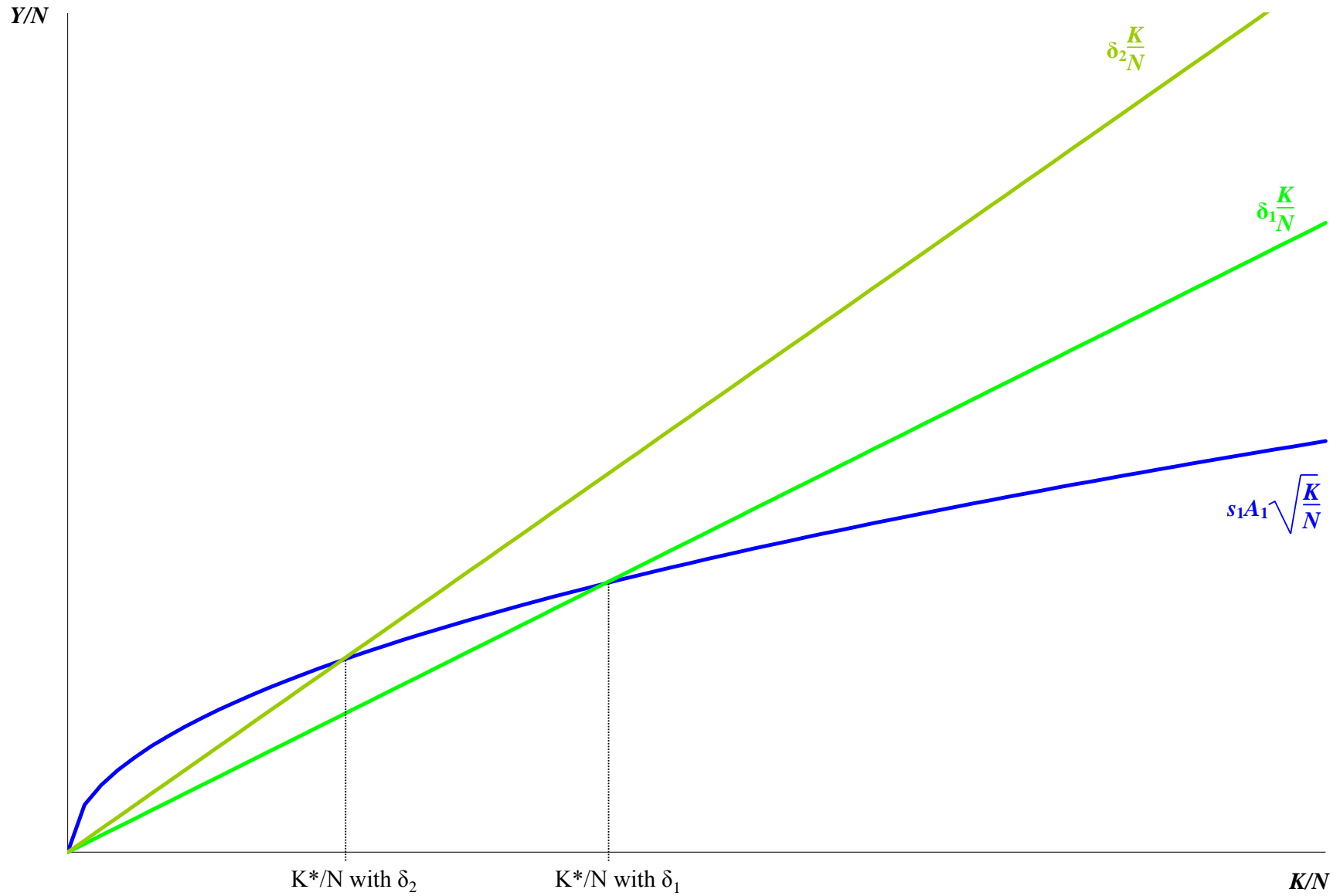


Figure 4

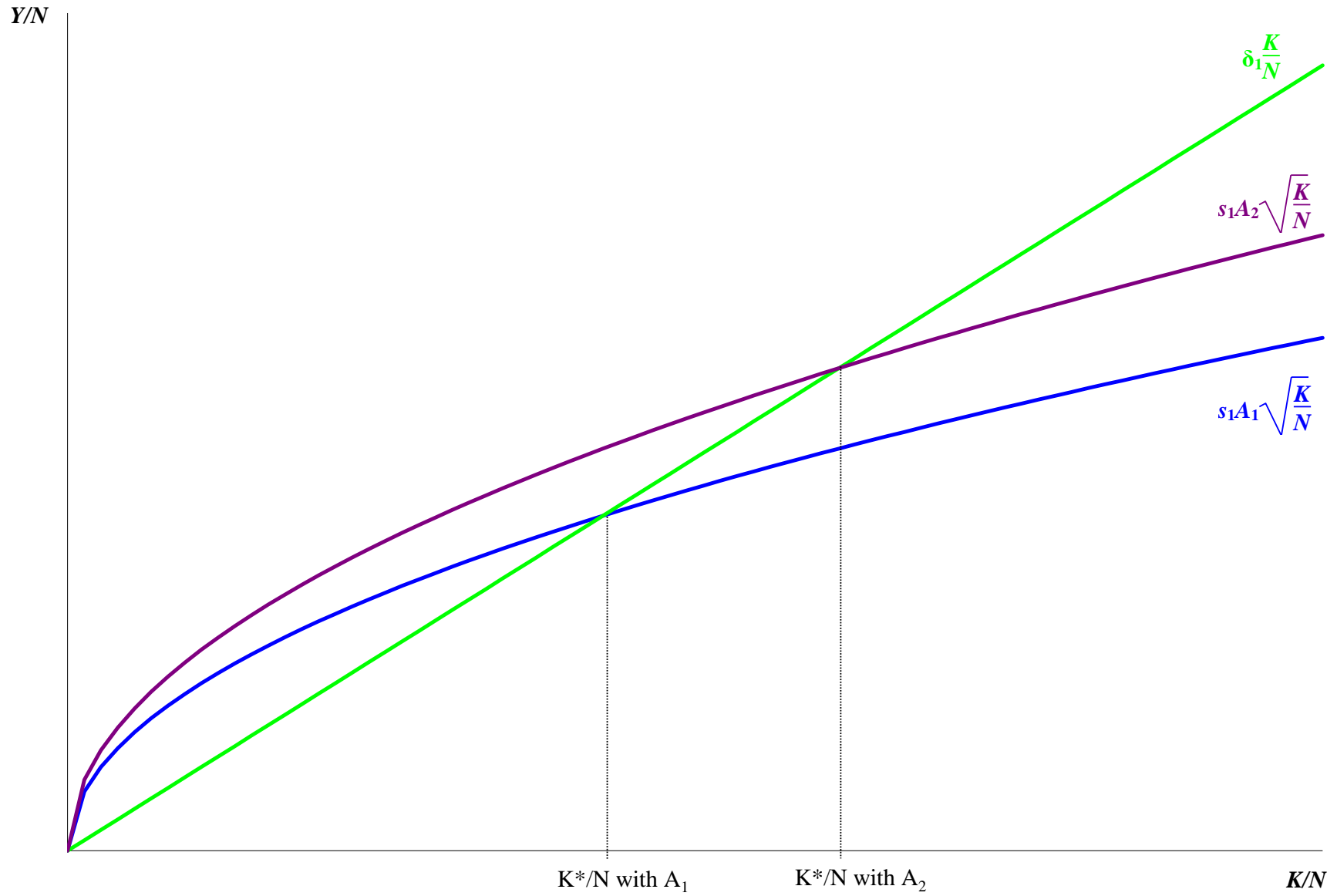
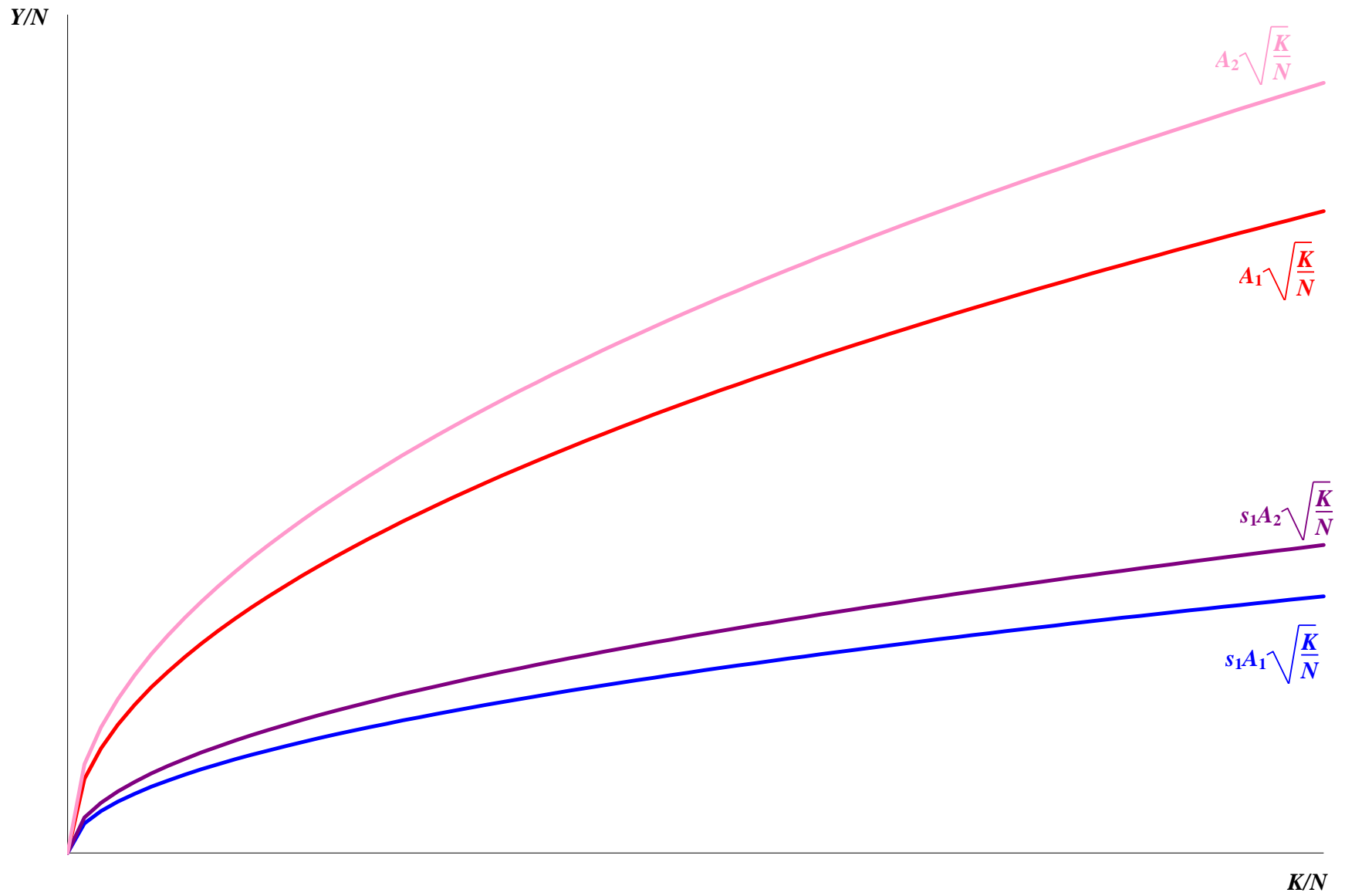


Figure 5



3. Consider a simple version of the Solow growth model, with no technology growth and no growth in employment. Suppose that the economy is described by the production function $Y = A\sqrt{K}\sqrt{N}$, where Y is output, K is the level of the capital stock, N is the (constant) number of workers, and A is the (constant) level of technology. Assume throughout that taxes and government expenditure are zero, and that the level of technology, A , is fixed at a value of 4.

a) Calculate the level of output corresponding to a capital stock of 250 units and employment of 250 workers. Calculate the level of output corresponding to a capital stock of 500 units and employment of 500 workers. Does this production function exhibit constant returns to scale? Explain.

answer:

$$Y(250,250) = 4\sqrt{250}\sqrt{250} = 4 \times 250 = 1000.$$

$$Y(500,500) = 4\sqrt{500}\sqrt{500} = 4 \times 500 = 2000.$$

Constant returns to scale requires that multiplying both inputs simultaneously by any arbitrary fixed proportion will lead to output also being multiplied by that same fixed proportion. In this case, doubling both inputs simultaneously leads to a doubling of output, i.e. everything has been multiplied by the fixed proportion 2. So yes, this production function does exhibit constant returns to scale.

b) Hold employment constant at 250 workers. Calculate the difference between the levels of output corresponding to capital stock levels of 250 and 300 units. Calculate the difference between the levels of output corresponding to capital stock levels of 500 and 550 units. Does this production function exhibit diminishing returns to capital? Explain.

answer:

$$Y(250,250) = 4\sqrt{250}\sqrt{250} = 4 \times 250 = 1000;$$

$$Y(300,250) = 4\sqrt{300}\sqrt{250} = 4 \times 17.3205 \times 15.8114 = 1095.45;$$

$$\text{difference} = 1095.45 - 1000 = 95.45.$$

$$Y(500,250) = 4\sqrt{500}\sqrt{250} = 4 \times 22.3607 \times 15.8114 = 1414.22;$$

$$Y(550,250) = 4\sqrt{550}\sqrt{250} = 4 \times 23.4521 \times 15.8114 = 1483.24;$$

$$\text{difference} = 1483.24 - 1414.22 = 69.02.$$

Diminishing marginal returns to capital requires that, with technology and labor held constant, a given increase in the capital stock will lead to a *smaller* increase in output the *larger* is the starting level of the capital stock. This is exactly what we see in this example: technology is held

constant at 4 throughout and labor is held constant at 250; the capital stock is increased by 50 units, first from a low starting point (250) and then again from a high starting point (500); and the resulting increase in output is smaller in the latter case (69 units compared to 95 units). So yes, this production function does exhibit diminishing returns to capital.

c) Assume that the depreciation rate is 10% (i.e. $\delta = 0.1$). Calculate the steady state capital-labor ratio and the steady state levels of output per worker and consumption per worker for saving rates of 25%, 50% and 75%. (You can use the expressions you derived in the first problem, but it would be good practice to derive them again from the capital accumulation equation and the specific production function of this problem. Using memorized expressions on tests and exams earns very little partial credit and isn't a good way to learn. To think visually about what is going on, think back to part b) of problem 2, perhaps adding the production function to the graph.)

answer:

I'll gloss over the derivations quickly, but I'll also reiterate the advice in the question: do the derivation again in detail for practice.

$$\frac{K_{t+1}}{N} - \frac{K_t}{N} = 4s\sqrt{\frac{K_t}{N}} - 0.1 \frac{K_t}{N}$$

In the steady state,

$$4s\sqrt{\frac{K^*}{N}} = 0.1 \frac{K^*}{N}$$

$$\frac{K^*}{N} = (4s/0.1)^2, \text{ so}$$

$$\frac{K^*}{N}(0.25) = (4 \times 0.25 / 0.1)^2 = (10)^2 = 100; \frac{K^*}{N}(0.5) = (20)^2 = 400; \frac{K^*}{N}(0.75) = (30)^2 = 900.$$

Steady state output per worker is given by

$$\frac{Y^*}{N} = 4\sqrt{\frac{K^*}{N}} = 4\sqrt{(4s/0.1)^2}, \text{ so}$$

$$\frac{Y^*}{N}(0.25) = 4\sqrt{(4 \times 0.25 / 0.1)^2} = 4\sqrt{(10)^2} = 40; \frac{Y^*}{N}(0.5) = 4 \times 20 = 80; \frac{Y^*}{N}(0.75) = 120.$$

Notice the effects of the curvature of the production function: an increase in the capital-labor ratio of 300 units (from 100 to 400) causes an increase in output per worker of 40 units (from 40 to 80); but in order to achieve a further increase in output per worker of 40 units (from 80 to 120), an increase in the capital-labor ratio of 500 units (from 400 to 900) is required.

Steady state consumption per worker is given by

$$\frac{C^*}{N} = (1 - s) \frac{Y^*}{N} = 4(1 - s) \sqrt{(4s/0.1)^2}, \text{ so}$$

$$\frac{C^*}{N}(0.25) = 4 \times (1 - 0.25) \times \sqrt{(10)^2} = 30; \quad \frac{C^*}{N}(0.5) = 4 \times (0.5) \times 20 = 40; \quad \frac{C^*}{N}(0.75) = 30.$$

This illustrates the hump-shaped relationship between consumption per worker in the steady state and the saving rate: steady state consumption per worker first rises with an increase in the saving rate, then falls as the saving rate continues to rise. At very high saving rates (i.e. 75% in this example), output per worker is indeed high, but the majority of it must be spent in the form of investment per worker to offset the high depreciation per worker associated with the large capital-labor ratio, leaving little remaining to be enjoyed in the form of consumption per worker. (We can't show this formally without more work, but it turns out that 50% is the "golden rule" saving rate in this example, i.e. the saving rate corresponding with the maximum possible steady state consumption per worker.)

You can check that everything adds up the way it should for any given saving rate. For example, with the 25% saving rate, we see that the steady state capital-labor ratio is 100 and the steady state level of output per worker is 40. The corresponding depreciation per worker is the depreciation rate of 10% times the capital-labor ratio, giving 10. Steady state investment per worker is the saving rate of 25% times output per worker, giving 10. Depreciation per worker and investment per worker are therefore equal at 10, which we would expect in the steady state. Steady state consumption per worker was found above to be 30, which is equal to output per worker (40) minus investment per worker (10), which should obviously be true from the way we derived consumption per worker.

d) Continue to assume that the depreciation rate is 10%. Suppose that the economy has been in the steady state corresponding to a saving rate of 50% for a number of periods. Then, in some period t , the saving rate suddenly and permanently falls to 25%. Calculate the capital-labor ratio in periods $t+1$, $t+2$ and $t+3$.

answer:

$$\frac{K_{t+1}}{N} = \frac{K_t}{N} + 4s \sqrt{\frac{K_t}{N}} - 0.1 \frac{K_t}{N}$$

This is the capital accumulation equation divided through by employment and with the production function plugged in. If you plug in a saving rate of 50% and a capital-labor ratio in period 0, for example, of 400, you will find that the capital-labor ratio in the following period 1 is also 400. This should be expected, because, as was found in the previous part, 400 is the steady state capital-labor ratio corresponding to a saving rate of 50%. What this part is asking you to imagine is that the economy has been proceeding from period to period in exactly this fashion, with some capital depreciating each period and some new capital being invested in each period, with the net result that the capital-labor ratio is staying fixed at 400 from period to

period. Then, suddenly, the saving rate falls to 25% and stays there. We know the new steady state capital-labor ratio that the economy will eventually converge to (100), and we characterized the convergence process (in reverse) in words in the second problem; in this example, we can illustrate the beginning of the convergence process numerically, which is what we're asked to do.

In period t , the saving rate changes (and hence, so does investment), but the capital stock does not adjust immediately, and so the capital-labor ratio remains at the original steady state level in period t . However, the lower saving and hence investment means that the capital-labor ratio will be lower in the *next* period, $t+1$:

$$\frac{K_{t+1}}{N} = 400 + 4 \times 0.25 \times \sqrt{400} - 0.1 \times 400 = 400 + 1 \times 20 - 40 = 380.$$

This lower capital-labor ratio in $t+1$ (lower than the previous level of 400, that is) will lead to output per worker and hence investment per worker levels lower than their period- t levels as well, which will lead to a yet lower capital-labor ratio in period $t+2$:

$$\frac{K_{t+2}}{N} = \frac{K_{t+1}}{N} + 4s\sqrt{\frac{K_{t+1}}{N}} - 0.1 \frac{K_{t+1}}{N}$$

$$\frac{K_{t+2}}{N} = 380 + 4 \times 0.25 \times \sqrt{380} - 0.1 \times 380 = 361.4936.$$

And so on in period $t+3$ and thereafter until the economy converges to the new steady state:

$$\frac{K_{t+3}}{N} = \frac{K_{t+2}}{N} + 4s\sqrt{\frac{K_{t+2}}{N}} - 0.1 \frac{K_{t+2}}{N}$$

$$\frac{K_{t+3}}{N} = 361.4936 + 4 \times 0.25 \times \sqrt{361.4936} - 0.1 \times 361.4936 = 344.3572.$$

The table on the next page shows how the capital-labor ratio, output per worker, investment per worker and depreciation per worker evolve over the several periods following the change in the saving rate, until the economy has converged all the way to the new steady state, where it will remain thereafter in the absence of any further shocks.

The table shows all periods up to $t+10$, but then only selected periods up to $t+175$. It's not at all clear that we can interpret a "period" in this model as a calendar year. Two centuries is a lot longer a time period than what we would typically want to consider as the "long run". On the other hand, the last hundred or so periods in this example are pretty much just cleaning up the decimal places as far as convergence goes. There is a discussion on page 228 of the textbook that touches on the duration of the convergence process, and our example largely matches that discussion: about two fifths of convergence in output per worker has occurred by period $t+10$, two thirds by $t+20$ and over 90% by $t+50$. The book interprets a period as a year in this model, and generally sticks to the notion of "the long run" encompassing up to a handful of decades.

Our example largely corresponds with this, and any discrepancies certainly don't detract from its illustrative value.

The point that was noted in the solution to part b) of the second problem should also be reiterated here. This model is not capable of accounting for the short-run expansionary effect of decreasing saving that we studied at the beginning of the semester. Again, we have to imagine that each "period" in this model encompasses medium-run adjustments following such short-run movements. This model can only describe what happens *across* these periods, and is not fine-tuned enough to describe what happens *within* one of these periods. So when the saving rate decreases in period t in this example, imagine that there is a brief expansion in output per worker to something higher than 80, but that the economy has adjusted back to output per worker of 80 by the end of period t , and then the long-run convergence cycle begins as we've discussed.

Finally, it should be noted that the specific saving rates you were asked to consider in this part are higher than what is typically observed in reality. It's not clear exactly how to measure a saving rate that corresponds precisely with our very simplified model, but it can be noted that few countries have an investment-to-GDP ratio greater than 25%. Again, this does not detract from the illustrative value of the example.

	K/N	Y/N	I/N	$\delta K/N$
t-3	400.0	80.0	40.0	40.0
t-2	400.0	80.0	40.0	40.0
t-1	400.0	80.0	40.0	40.0
t	400.0	80.0	20.0	40.0
t+1	380.0	78.0	19.5	38.0
t+2	361.5	76.1	19.0	36.1
t+3	344.4	74.2	18.6	34.4
t+4	328.5	72.5	18.1	32.8
t+5	313.8	70.9	17.7	31.4
t+6	300.1	69.3	17.3	30.0
t+7	287.4	67.8	17.0	28.7
t+8	275.6	66.4	16.6	27.6
t+9	264.7	65.1	16.3	26.5
t+10	254.5	63.8	16.0	25.4
t+20	183.6	54.2	13.5	18.4
t+50	115.7	43.0	10.8	11.6
t+100	101.2	40.2	10.1	10.1
t+150	100.1	40.0	10.0	10.0
t+175	100.0	40.0	10.0	10.0