Practice Problem Set #6 – Solutions
EC202C1, Spring 2012, Jeremy Smith

1. Consider the full Solow growth model with exogenous technological progress and employment growth. The economy is described by the production function $Y = \sqrt{K}\sqrt{AN}$. Technology, A, grows at the fixed percentage rate g_A per period, and employment, N, grows at the fixed percentage rate g_N per period. The capital stock, K, depreciates at the fixed rate δ per period, and the economy saves a fixed proportion, s, of output, Y, per period. Assume throughout that taxes and government expenditure are zero.

a) Convert the production function to intensive form (i.e. find an expression for output per effective worker).

answer:

$$Y = \sqrt{K}\sqrt{AN}$$

$$\frac{Y}{AN} = \frac{\sqrt{K}\sqrt{AN}}{AN}$$ [divided both sides by AN]

$$\frac{Y}{AN} = \frac{\sqrt{K}}{\sqrt{AN}}$$ [because $\sqrt{AN}/AN = 1/\sqrt{AN}$]

$$\frac{Y}{AN} = \sqrt{\frac{K}{AN}}.$$ [This is what we refer to generally as $f\left(\frac{K}{AN}\right)$.]

b) Write down the condition that must hold for the change in capital per effective worker between years t and t+1 to be zero and hence for the economy to be in a steady state. (You do not need to derive this from the capital accumulation equation as we did in class. Instead, reason your way towards it, as is done at the bottom of page 241 in the custom-printed version of the textbook. You should understand this reasoning, and write it down in your own words.)

answer:

In the steady state, capital per effective worker will be constant. We therefore need to figure out how fast the capital stock needs to change from period to period (i.e. how much needs to be invested in each period) to keep this ratio constant. If investment were zero in every period, the ratio of capital to effective workers would be getting smaller over time for three reasons: the capital stock would be depreciating each period, so the numerator would be getting smaller; technology would be growing in each period, so the denominator would be getting bigger; and the number of workers would be growing in each period, so again the denominator would be getting bigger. Therefore, to keep the ratio constant from period to period, investment must be enough to offset all three of these forces. In proportion to the existing level of the capital stock in any period t: depreciation reduces the ratio K/AN by the factor δ; technology growth reduces the ratio K/AN by the factor g_A; and growth in the number of workers reduces the ratio K/AN by the factor g_N. Thus, to keep the ratio K/AN constant, investment in period t must add enough to compensate for all three of these factors, also in proportion to the existing level of the capital stock in period t. In equation form, this requires that

$$I_t = (\delta + g_A + g_N)K_t$$

to keep capital per effective worker constant across periods t and t+1.

(If this loose reasoning is not satisfying for you, there is a way to derive this condition more rigorously. Start by imposing the equality of capital per effective worker across periods, which is the very situation we want the investment condition to support. That is, in equation form, just set $K_{t+1}/A_{t+1}N_{t+1} = K_t/A_tN_t$. Now note that $K_{t+1} = K_t - \delta K_t + I_t$ by the standard capital accumulation equation; so the right-hand side of this equation can be substituted in for K_{t+1} on the left-hand side of the capital-per-effective-worker equality. Then just solve this equality for I_t. To clean up the resulting investment condition and get it in the exact form as above will involve applying the expressions for the growth of technology and workers across periods and the approximation that two growth rates multiplied by each other is virtually zero.)

To be clearer, I should have phrased this question as "Write down the condition ON INVESTMENT that must hold for the change in capital per effective worker between years t and t+1 to be zero…" Just imposing the equality of the capital-per-effective-worker ratio across periods and not doing anything with it would not have been enough. What I was trying to elicit was the intuition that, if that equality is to hold, investment (i.e. the flow that positively affects the level of the capital stock and thus the only thing that can make the ratio K/AN higher) must be enough to compensate for all three factors (i.e. depreciation, technology growth, and employment growth) that push the ratio K/AN down.

c) Find expressions for the steady state levels of capital per effective worker and output per effective worker. Show your work.

answer:

Start with the condition just derived, and divide by effective workers:

$$\frac{I_t}{A_tN_t} = (\delta + g_A + g_N)\frac{K_t}{A_tN_t}.$$

Now, note that investment equals saving in equilibrium as usual, and that investment is the saving rate times output in any period:

$$\frac{sY_t}{A_tN_t} = (\delta + g_A + g_N)\frac{K_t}{A_tN_t}.$$

Output per effective worker on the left can now be replaced with the intensive form of the production function:

$$s\sqrt{\frac{K_t}{A_tN_t}} = (\delta + g_A + g_N)\,\frac{K_t}{A_tN_t}\,.$$

Finally, we can drop time subscripts and replace them with our usual asterisk notation for the steady state, and solve:

$$s\sqrt{\left(\frac{K}{AN}\right)^*} = (\delta + g_A + g_N)\left(\frac{K}{AN}\right)^*$$

$$\frac{s}{(\delta + g_A + g_N)} = \sqrt{\left(\frac{K}{AN}\right)^*}$$ [divide both sides by $(\delta + g_A + g_N)\sqrt{\left(\frac{K}{AN}\right)^*}$ and note that $a / \sqrt{a} = \sqrt{a}$ for any arbitrary variable a]

$$\left(\frac{K}{AN}\right)^* = \left(\frac{s}{(\delta + g_A + g_N)}\right)^2.$$ [square both sides, and rearrange]

To find the corresponding level of output per effective worker in the steady state, plug this into the intensive form of the production function:

$$\frac{Y}{AN} = \sqrt{\frac{K}{AN}}$$

$$\left(\frac{Y}{AN}\right)^* = \sqrt{\left(\frac{K}{AN}\right)^*}$$

$$\left(\frac{Y}{AN}\right)^* = \sqrt{\left(\frac{s}{(\delta + g_A + g_N)}\right)^2} = \frac{s}{(\delta + g_A + g_N)}\,.$$

d) Find the steady state growth rates of output per effective worker, output per worker and output. Also find the steady state growth rate of consumption per worker. Show and explain your work.

answer:

We have just found that output per effective worker is constant in the steady state (because the expression is fully comprised of fixed exogenous parameters). Therefore, in the steady state,

$g_{Y/AN}* = 0.$

Using our rules for growth rates, we also know that

$$g_{Y/AN} = g_Y - g_A - g_N$$

in general, so, combining these two results,

$g_Y^* - g_A - g_N = 0$,

from which we can derive the other results. For output per worker,

$g_{Y/N} = g_Y - g_N$

by definition, and in the steady state specifically,

$g_{Y/N}^* = g_A$

because $g_Y^* - g_A - g_N = 0$ (which we just found is true in the steady state) can be rearranged to $g_Y^* - g_N = g_A$. So, in the steady state, output per worker grows at the rate of growth of technology.

Likewise, for output, in the steady state

$g_Y^* = g_A + g_N$

because $g_Y^* - g_A - g_N = 0$ (which we just found is true in the steady state) can be rearranged to $g_Y^* = g_N + g_A$. So, in the steady state, output grows at the rate of growth of technology plus the rate of growth of employment.

For consumption per worker, we need to start with the condition for goods market equilibrium:

$Y = C + I + G$

$Y = C + I$ [because government expenditure is zero by assumption]

$C = Y - I$ [rearranging]

$C = Y - sY$ [because investment equals saving in equilibrium, and saving is the saving rate times output as usual]

$\frac{C}{N} = (1 - s)\frac{Y}{N}$. [simplifying and dividing both sides by N]

Instead of dividing by employment at this last step, we could have divided by effective workers and solved for a symbolic expression for steady state consumption per effective worker. Then, noting that this expression is constant, we could have proceeded as we did above with output per effective worker. However, we can actually proceed directly with growth rates at this last step:

$g_{C/N} = g_{(1-s)} + g_{Y/N}$ [the growth rate of a product is the sum of the growth rates of the individual components]

$g_{C/N} = 0 + g_{Y/N}$ [the saving rate is constant]

$g_{C/N} = g_{Y/N}$.

That is, consumption per worker grows at the same rate as output per worker in this model. Therefore, in the steady state,

$g_{C/N}* = g_A$

because this is the growth rate of output per worker in the steady state, as we have already shown. So, in the steady state, consumption per worker grows at the rate of growth of technology.

e) Find the values of the steady state level of capital per effective worker and the steady state growth rate of output per worker if the saving rate is 16%, the depreciation rate is 10%, employment growth is 2% and technology growth is 4%.

answer:

$$\left(\frac{K}{AN}\right)^* = \left(\frac{s}{(\delta + g_A + g_N)}\right)^2 = \left(\frac{0.16}{(0.1 + 0.02 + 0.04)}\right)^2 = \left(\frac{0.16}{(0.16)}\right)^2 = 1^2 = 1.$$

In the steady state,

$g_{Y/N}* = g_A = 4\%$.

f) Find the values of the steady state level of capital per effective worker and the steady state growth rate of output per worker if the saving rate is 14%, the depreciation rate is 5%, employment growth is 3% and technology growth is 6%.

answer:

$$\left(\frac{K}{AN}\right)^* = \left(\frac{s}{(\delta + g_A + g_N)}\right)^2 = \left(\frac{0.14}{(0.05 + 0.03 + 0.06)}\right)^2 = \left(\frac{0.14}{(0.14)}\right)^2 = 1^2 = 1.$$

In the steady state,

$g_{Y/N}* = g_A = 6\%$.

g) Comparing parts e) and f), does it appear that, other things equal, the level of capital per effective worker has any relevance for the average standard of living in the long run?

answer:

The economies in the previous two parts both have capital per effective worker levels of 1 in their respective steady states. But the underlying variables describe two very different worlds. The point is that all of the per-effective-worker variables are abstract quantities that bear little relation to things we might care about in reality. The only reason we divide through by effective workers is to help us find the steady state and to analyze it tractably and coherently. But once we derive the expression and look at it statically on a graph, the real interest lies in figuring out the steady state growth paths of variables of more concrete interest.

One such variable of concrete interest is output per worker, which is a common measure of the average "standard of living". As has been argued previously, a better one is consumption per worker, because it tells us what we actually get to enjoy out of what we produce, net of what we spend just to maintain the capital stock. But as we saw earlier, output per worker and consumption per worker grow at the same rate in the steady state in this model. The economy in part f) will enjoy growth of 6% per year in consumption per worker in the long run, while the economy in part e) will enjoy growth of 4% per year in consumption per worker in the long run. This is a big difference. Stretched over decades, a difference in growth rates of two percentage points will lead to a huge separation in levels of the standard of living between the two economies. In the meantime, the level of capital per effective worker will be at 1 in both economies the whole time, conveying basically zero information about the standard of living.

Of course, it is the *level* of consumption per worker at a point in time that a society might be more worried about, rather than just its growth rate. It's certainly possible that the economy in part f) started with a much lower level of consumption per worker than the economy in part e) when each first reached its steady state. But given enough time, compound growth is an enormously powerful thing. Even if the second economy had started with half the level of consumption per worker as the first economy, the second economy will take the lead within four decades. (Don't worry about how I calculated this. The point is, small differences in growth rates matter a lot, while the level of capital per effective worker doesn't matter at all from a practical standpoint.)

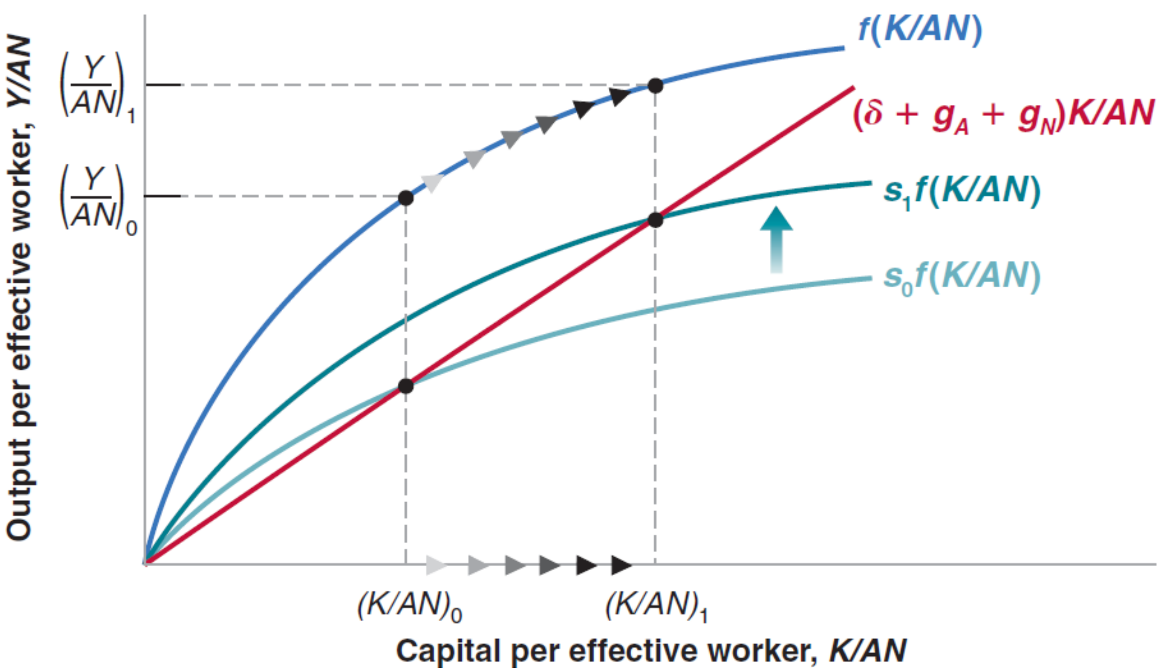
(For an interesting illustration of growth rates versus levels, consider the case of the United States and China. According to Angus Maddison's *The World Economy: Historical Statistics* (OECD: 2003) – a truly amazing resource – China had a level of GDP per capita of 3583 in 2001, compared to 27948 for the United States – in internationally comparable physical units that we don't need to get into the technicalities of. But between 1995 and 2001, the geometric average annual growth rate of GDP per capita was 5.14% in China and 2.23% in the United States. If this growth differential persists, China, starting from about an eighth the level of the standard of living in 2001, will have caught up to the US standard of living level – which itself will be much higher than it is now – by 2080 or earlier. Of course, the danger in taking these kinds of fun examples too literally is that the growth rates and much else besides will certainly change over such a long time horizon, sometimes in ways we could try to predict, but often not.)

h) Suppose that the economy is in the steady state corresponding to the parameter values in part e). Then, the saving rate suddenly and permanently jumps to 32%. Find the new steady state level of capital per effective worker corresponding to the higher saving rate. Draw and carefully label a graph illustrating this situation, showing the levels of capital per effective worker and output per effective worker in the old steady state and in the new steady state. Describe briefly and without doing any calculations how the economy will converge to the new steady state. What happens to the growth rate of output per worker as the economy moves from the old steady state to the new steady state?

answer:

$$\left(\frac{K}{AN}\right)^* = \left(\frac{s}{(\delta + g_A + g_N)}\right)^2 = \left(\frac{0.32}{(0.1 + 0.02 + 0.04)}\right)^2 = \left(\frac{0.32}{(0.16)}\right)^2 = 2^2 = 4.$$

The following graph reproduces Figure 3 from Chapter 12 of the textbook (the top of page 244 in the custom-printed version). It conveys all of the relevant information.



In this specific example, $s_0 = 0.16$, $s_1 = 0.32$, $f\left(\frac{K}{AN}\right) = \sqrt{\frac{K}{AN}}$, $\left(\frac{K}{AN}\right)_0 = \left(\frac{K}{AN}\right)^*[0.16] = 1$,

$\left(\frac{K}{AN}\right)_1 = \left(\frac{K}{AN}\right)^*[0.32] = 4$, $\left(\frac{Y}{AN}\right)_0 = \left(\frac{Y}{AN}\right)^*[0.16] = 1$, $\left(\frac{Y}{AN}\right)_1 = \left(\frac{Y}{AN}\right)^*[0.32] = 2$, and

$(\delta + g_A + g_N) = 0.16$. It's not strictly necessary to replace the general notation with the specific information for the graph to be acceptable, as long as the steady states are clearly marked as such and the investment-per-effective-worker functions, "necessary investment"-per-effective-worker line and intensive production function (i.e. the output-per-effective-worker function) are sufficiently demarcated and named. But it's a good idea to be as specific as possible when you're practicing these types of examples, to make sure that you're not just learning by rote, which isn't a good way to prepare for exams.

The new steady state will entail a higher level of capital per effective worker compared with the original steady state. If the economy starts in the original steady state, investment in a given period in the steady state will, by definition, be just enough to offset deterioration in capital per effective worker due to depreciation, technology growth and employment growth, and hence to keep capital per effective worker constant. In the period in which the saving rate increases, investment per effective worker will immediately increase from its steady-state level. (Capital per effective worker and hence output per effective worker will stay constant in the instant that the saving rate increases. But investment per effective worker, which is given by the saving rate times output per effective worker, will be higher due to the higher saving rate.) The higher investment per effective worker but unchanged level of "necessary investment" (or "required investment") per effective worker (unchanged because, again, the capital stock does not change immediately, and it is also assumed that technology and employment growth are only realized at the end of the period) will lead to a higher level of capital per effective worker in the next period, and hence a higher level of output per effective worker. The higher level of output per effective worker will lead to yet higher investment per effective worker, which in turn will lead to yet higher levels of capital per effective worker and output per effective worker, again leading to an increase in investment per effective worker. At the same time, necessary investment per effective worker is increasing as the level of capital per effective worker increases. Indeed, the increase in investment per effective worker across successive periods will be slightly smaller than the increase in necessary investment per effective worker, due to the curvature of the intensive production function and hence the investment-per-effective-worker function (reflecting diminishing marginal returns to capital) versus the linearity of the necessary investment-per-effective-worker function. This cycle – from investment to capital to output and back to investment – will hence proceed in smaller and smaller steps over time, and will eventually conclude when the size of the step reaches zero, at which point the economy will have converged to its new steady state. In this new long run equilibrium, the higher steady state level of investment per effective worker will again be exactly enough to offset the higher steady state level of necessary investment per effective worker and hence keep capital per effective worker constant at its new, higher steady state level from period to period.

In the initial steady state, output per worker will be growing at the rate of growth of technology (as usual in the steady state), which is 4% in this example. And in the new steady state, output per worker will again be growing at 4%. The rate of technology growth does not change, and steady state growth in output per worker is determined solely by technology growth, so we know that output per worker will be growing at 4% in both steady states.

What happens during the convergence process is more interesting. Because output per effective worker is constant in the steady state, we know that output must be growing at the rate of growth of technology plus the rate of growth of employment in the steady state. (This comes directly from our growth rate rules and the reasoning applied in part d) above.) But because output per effective worker is growing at a positive rate as the economy converges to the new steady state (i.e. it increases from period to period, as just described above), it must be the case that output is growing *more rapidly than* the sum of technology and employment growth (remember: both growth rates are held fixed throughout) during the convergence process. And as output per effective worker converges to its higher constant level in the new steady state (i.e. its growth converges back to zero), it must be that the rate of growth of output is slowly decreasing back to the rate of growth of technology plus the rate of growth of employment. All of these statements can easily be converted to terms of output per worker rather than output with the usual growth rate rules.

So, in sum, the growth rate of output per worker is 4% in the initial steady state, jumps to something higher the period after the saving rate changes, continues to be above 4% for several periods as the economy converges to the new steady state, but slowly decreases period by period, and eventually returns to 4% when the new steady state is obtained.